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A unitarity analysis on the I=0 d wave $\pi\pi$ scattering amplitude

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Abstract: We study I=0 d wave $\pi\pi$ scattering phase shift using a proper unitarization approach. It is verified that the $f_2(1270)$ resonance corresponds to a twin pole structure: one on the second sheet, another one on the third sheet. Also we find fair agreement on $f_2(1270)$ pole's mass and width between our results and the PDG values. Besides, our analysis reveals the existence of a virtual state in this channel. Its pole location is determined by fitting experimental data and it is found to be in good agreement with the prediction from chiral perturbation theory. Our analysis demonstrates that partial wave amplitudes calculated in chiral perturbation theory are reliable in the small $|s|$ region, contrary to some unitarization models.

Keywords: Chiral Lagrangians.

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In a series of recent publications we have developed a new dispersion approach incorporating unitarity and chiral symmetry, to study scatterings between pseudo-Goldstone bosons. The approach is found to be particularly useful in revealing low energy pole structure of the scattering amplitudes. It is demonstrated that the σ meson is essential to adjust chiral perturbation theory (χPT) to experimental data [1]. Combining with crossing symmetry, the mass and width of the σ meson are carefully determined [2] and are found in nice agreement with the Roy equation analysis [3]. By analyzing the LASS data [4] it is also demonstrated that the κ resonance must exist in I=0,J=1/2 channel of πK scatterings if χPT prediction on the scattering length in the corresponding channel is acceptable [5] It is also found that, interestingly, in I=2,J=0 channel of $\pi\pi$ scattering, a virtual state pole is needed in order to fit the experimental data and to correctly reproduce the chiral prediction on the scattering length [2]. The existence of such a virtual state pole is actually a prediction of current algebra and χPT [6], hence demonstrating that χPT provides a consistent description to the scattering T matrix in the small $|s|$ region. In this note we will further investigate low energy I=0 d-wave $\pi\pi$ scatterings, using the phase shift data from ref. [12]. The results provide further evidences to support χPT predictions in the small $|s|$ region.

We start from a proper parametrization form, for the partial wave S matrix [2, 7]:

$$S^{\rm phy} = \prod_i S^{p_i} \cdot S^{\rm cut}\,, \tag{1}$$

where S^{p_i} are the simplest S matrices characterizing the isolated singularities of $S^{\rm phy}$ on the second sheet in the absence of bound states, which are:

1. For a virtual state pole located at $s = s_0$ $(0 < s_0 < 4m_\pi^2)$,

$$S(s) = \frac{1 + i\rho(s)a}{1 - i\rho(s)a}\,, \tag{2}$$

   and a is the scattering length:

$$a = \sqrt{\frac{s_0}{4m_\pi^2 - s_0}}\,. \tag{3}$$

   The kinematic factor $\rho(s) = \frac{2k(s)}{\sqrt{s}} = \sqrt{\frac{s-4m_\pi^2}{s}}$.

2. For a resonance located at z_0 (and z_0^*) on the second sheet, the S matrix can be written as,

$$S^R(s) = \frac{M^2[z_0] - s + i\rho(s)sG[z_0]}{M^2[z_0] - s - i\rho(s)sG[z_0]}\,, \tag{4}$$

   where

$$\begin{aligned} M^2[z_0] &= \mathrm{Re}[z_0] + \mathrm{Im}[z_0]\frac{\mathrm{Im}[\sqrt{z_0(z_0-4m_\pi^2)}]}{\mathrm{Re}[\sqrt{z_0(z_0-4m_\pi^2)}]}\,, \\ G[z_0] &= \frac{\mathrm{Im}[z_0]}{\mathrm{Re}[\sqrt{z_0(z_0-4m_\pi^2)}]}\,. \end{aligned} \tag{5}$$

According to eq. (1), each pole's contribution to the scattering phase shift, denoted as $\delta_{p_i}(s)$, is additive and easily calculable using eqs. (2) and (4). In eq. (1) $S^{\rm cut}$ can be parameterized in the following simple form,

$$S^{\rm cut} = e^{2i\rho f(s)}\,, \tag{6}$$

$$f(s) \equiv f_L(s) + f_R(s) = \frac{s}{\pi}\int_L \frac{{\rm Im_L}f(s')}{(s'-s)s'}ds' + \frac{s}{\pi}\int_R \frac{{\rm Im_R}f(s')}{(s'-s)s'}ds'\,, \tag{7}$$

where $L = (-\infty, 0]$ denotes the left hand cut on the negative real axis, R denotes cuts from the first inelastic threshold to ∞, and the dispersion integral is free from subtraction constants. The discontinuity of $f(s)$ on left or right hand cut is expressed as

$${\rm Im}_{L,R}f(s) = -\frac{1}{2\rho(s)}\log|S^{\rm phy}(s)|\ . \tag{8}$$

The above parametrization is, however, not directly applicable to the study of d–wave scatterings since the following threshold constraint has not yet been taken into account:

$${\rm Re}T_{IJ}(s) = q^{2J}[a_J^I + b_J^I q^2 + O(q^4)]\,, \qquad (q = \frac{1}{2}\sqrt{s-4m_\pi^2}\,)\,, \tag{9}$$

To remedy this we first recast eq. (7) into a thrice subtracted form with a once-subtraction at $s = 0$, and a twice-subtraction at physical threshold $4m_\pi^2$:

$$f(s) = s\left(-f'(4m_\pi^2) + \frac{f(4m_\pi^2)}{2\,{m_\pi}^2}\right) + s^2\left(\frac{-f(4m_\pi^2)}{16\,{m_\pi}^4} + \frac{f'(4m_\pi^2)}{4\,{m_\pi}^2}\right) + \\ + \frac{s(s-4m_\pi^2)^2}{\pi}\int_{L+R}\frac{{\rm Im}f(s')}{(s'-s)s'(s'-4m_\pi^2)^2}ds'\,, \tag{10}$$

where the two subtraction constants, $f(4m_\pi^2)$ and $f'(4m_\pi^2)$ are not free, they are correlated to pole parameters as dictated by eq. (9):

$$f(4m_\pi^2) = \sum_i f_{p_i}\,,$$
$$f'(4m_\pi^2) = \sum_i f'_{p_i}\,, \tag{11}$$

where

$$f_r = \frac{4\,G\,{m_\pi}^2}{4\,{m_\pi}^2 - M^2}\,,$$
$$f'_r = -\frac{G\,\left(16\,G^2\,{m_\pi}^4 + 12\,{m_\pi}^2\,M^2 - 3\,M^4\right)}{3\,\left(4\,{m_\pi}^2 - M^2\right)^3} \tag{12}$$

for a resonance, and

$$f_v = -a\,, \qquad f'_v = \frac{a^3}{12\,{m_\pi}^2} \tag{13}$$

for a virtual state. Since the threshold constraint must be exactly obeyed, we in the following will make use of eq. (11) to replace $f(4m_\pi^2)$ and $f'(4m_\pi^2)$ in eq. (10) by corresponding pole parameters.

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According to eqs. (1), (6) and (7), the phase shift from partial wave S matrices for elastic $\pi\pi$ scatterings consist of contributions from second-sheet poles, left-hand cut integral and right-hand cut integral:

$$\delta(s) = \sum_i \delta^{\rm II}_{p_i}(s) + \rho(s) f(s)\,. \tag{14}$$

The right-hand cut represents effects from inelastic thresholds and poles from higher sheets. We have: [8]

$$\begin{aligned} {\rm Im}_R f(s) &= -\frac{1}{2\rho(s)} \log|S^{\rm phy}(s)| \\ &= -\frac{1}{4\rho} \log\left[1 - 4\rho{\rm Im}_R T + 4\rho^2|T(s)|^2\right] \\ &= -\frac{1}{4\rho} \log\left[1 - 4\rho\left(\sum_{n\neq 1}\rho_n|T_{1n}(s)|^2 + \cdots\right)\right]. \end{aligned} \tag{15}$$

As an approximation, when only poles are considered, we have the following parametrization for inelastic amplitudes [9]:

$$T_{1n} = \frac{1}{\sqrt{\rho_1(s)\rho_n(s)}} \sum_r \frac{\mathcal{M}_r(\Gamma_{r1}\Gamma_{rn})^{\frac{1}{2}}}{\mathcal{M}_r^2 - s - i\mathcal{M}_r\Gamma_r}\,, \tag{16}$$

with Γ_{rn} the partial width into channel n, Γ_r the total width of resonance r, and $\rho_n(s)$ represents the kinematic factor of n-th channel. Especially in above (and hereafter) subscript 1 denotes the 2π channel, and here R starts from 4π threshold. It should be stressed that the eq. (16) only works for narrow resonances which is only of limited usage, and furthermore, the narrow width approximation only works when the energy region is far from any physical threshold.

For the parametrization scheme described above, it should be realized that isolated singularities in a couple channel S matrix can play rather different roles depending on which sheet they locate. It is therefore necessary to elaborate more about the relation between a physical resonance and its pole structure in the couple channel situation. Taking the well established $f_2(1270)$ I=0 2^{++} resonance [10] for example, the PDG value of its mass and width are $M = 1275.4 \pm 1.1\,$MeV and $\Gamma = 185.2^{+3.1}_{-2.5}\,$MeV. The particle decays predominantly into $\pi\pi$ with a branching ratio ${\rm Br}(\pi\pi) = 84.8\%$. The second largest branching ratio is the 4π mode, ${\rm Br}(4\pi) = 10.2\%$ with sizable error bars. The $\bar{K}K$ mode is less important, with ${\rm Br}(\bar{K}K) = 4.6\%$. Most of the experimental results listed in ref. [10] are from production experiments using a standard Breit-Wigner parametrization form for the d-wave, which means the detected $f_2(1270)$ resonance is on the sheet smoothly connected to the upper edge of the unitarity cut (i.e., the place where experimentalists perform experiments). In a more formal language, under for example the two channel approximation, the experimentally established resonance is a 3rd sheet pole. However, in the present situation the picture of a couple channel resonance is more complicated. Indeed, phase shift data in the I,J=0,2 channel exhibit a typical resonance structure with a mass around 1270 MeV, as shown in figure 2. Nevertheless it is easy to understand from eq. (1) that the jump of the scattering phase shift at around 1270 MeV is provided by a second sheet pole. In the present scheme, the expression for the phase shift of the elastic scattering amplitude measured by experiments in the in-elastic region, according to eq. (14), reads as

$$\delta(s) = \sum_i \delta^{\mathrm{II}}_{p_i} + \rho_1 f_L(s) + \frac{s\rho_1(s)}{\pi}\mathcal{P}\int_R \frac{\mathrm{Im}_{\mathrm{R}} f(s')}{(s'-s)s'}ds' \,, \qquad (s > 16m_\pi^2)\,, \tag{17}$$

where $\mathcal{P}$ stands for principal value. The inelasticity parameter obeys the following expression,

$$\eta(s) = \exp[-2\rho_1(s)\mathrm{Im}_R f(s)]\ . \tag{18}$$

From eqs. (15) and (16) we know that physically observed narrow states in production experiments contribute to the inelasticity parameter. Meanwhile it can be verified that their contributions to the phase shift of the elastic amplitude are small kinks, rather than rapid jumps of approximately 180° (see figure 1 for illustration). In other words, if we could have accurate data on phase shift and especially inelasticity, we would be able to get more information on poles on different sheets.[1] Nevertheless in practice the data are usually not good enough to achieve such a goal.

The left hand integral appeared in eq. (10) converges very rapidly if we approximate S^{phy} in the integrand by $S^{\chi PT}$. It has been shown that such an approximation provides a satisfactory description to the data at qualitative level in exotic channels, and also provides at least a self-consistent description in non-exotic channels [2, 5, 1]. Therefore we in this paper will also use χPT result to evaluate the left hand integral. Owing to the thrice-subtraction, the cutoff parameter used when evaluating the left hand integral has only a tiny influence, and increasing the cutoff parameter will slightly improve the fit quality. Thus the cutoff parameter is practically fixed at ∞ and and is not treated as a free parameter.

The integral on the r.h.s. of eq. (10) behaves as $O(s^2)$ when s is large. The coefficient of the $O(s^2)$ term has a negative sign and it has a quite large effect at large s (for example, when $s > 1.4\,\mathrm{GeV}$). Thus it is demanded that the polynomial on the r.h.s. of eq. (10) be large enough to counteract the large negative contribution from the integral when s is large. It can be shown that the second sheet pole $f_2(1270)$, at around $M^2 \sim 1.63\,\mathrm{GeV}^2$ and $G \sim 0.13$, provides a negative contribution to the coefficient of s^2 term. That means contributions from other poles are necessary. A virtual state or a resonance located in some special region can offer a positive term to the coefficient. For the latter case it can be verified numerically that, if denoting the 2nd sheet resonance pole position as z_0 on the complex s plane, then $\mathrm{Re}[z_0] < 4m_\pi^2$. On the other hand, a bound state will offer a negative term, and a third sheet pole contributes through the right hand cut integral and it also provides a negative coefficient. Therefore at least one virtual state or one peculiar second sheet resonance should exist in the I=0 d-wave amplitude. However numerical analysis reveals that the latter case is not stable, the additional resonance will collapse and convert to two virtual states. Furthermore, one of the virtual state moves towards $s = 0$ (and hence

[1] More precisely speaking, under present scheme we can at best determine the partial decay width Γ_1 and the total decay width Γ_{tot} but no more.

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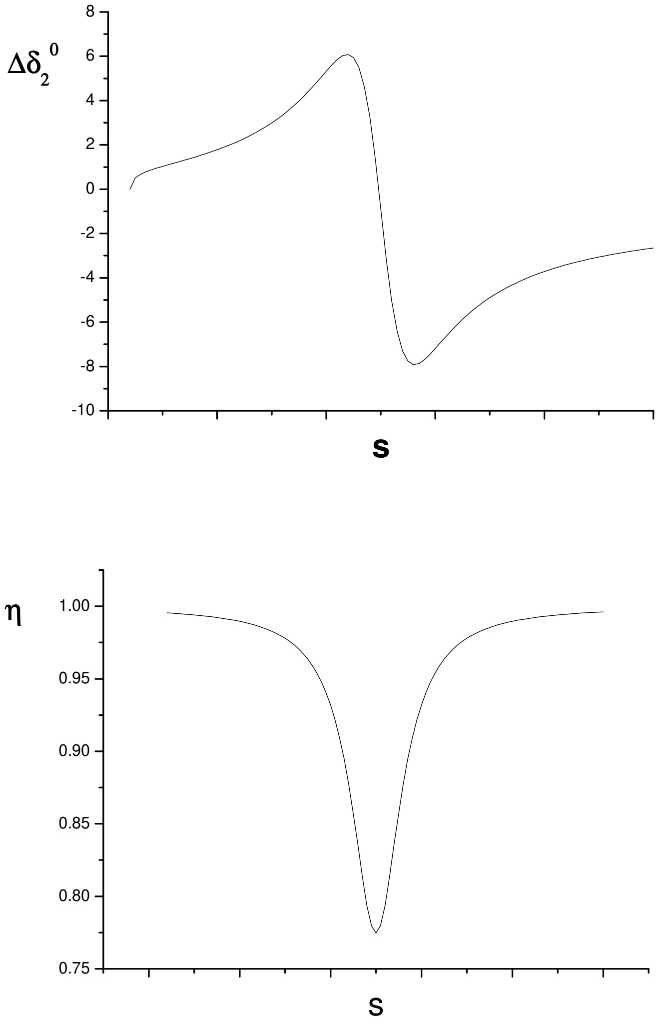

Figure 1: An illustrative example for a single resonance's contribution (hence without interference between different resonances) to the phase shift and the inelasticity parameter above the in-elasticity threshold, drawn using eqs. (17) and (18).

has a vanishing effect) and another remains very close to the location of the virtual pole when only the latter is used in the fit. Therefore the conclusion is that a virtual state pole is needed in order to correctly describe the data. This observation is remarkable, since the virtual pole's contribution to the phase shift is very small and hence is hard to be directly seen from the data. On the other hand the d-wave two-loop χPT amplitude does predict a virtual state pole at about 1.95 MeV, and the one loop amplitude at 1.52 MeV using the low energy constants provided by ref. [11].

Besides the $f_2(1270)$ resonance as described earlier, ref. [10] also provides several other 2^{++} resonances below 1.7 GeV: $f_2(1430)$, $f_2'(1525)$, $f_2(1565)$, except for the $f_2(1270)$ state. In the present approach they are all contained in the right hand integral and contribute to

$M_{\pi\pi}$ (MeV)	400	600	800	1000	1200	1400	1600
δ_2^0	0.125	1.013	2.914	10.34	44.66	140.83	157.85
$\triangle\delta_2^0$	0.024	0.088	0.146	0.30	0.88	2.22	8.74
η		0.968	0.955	0.923	0.790	0.760	0.859
$\triangle\eta$		0.011	0.013	0.019	0.028	0.044	0.064

Table 1: Phase shift value for the I=0,J=2 $\pi\pi$ scattering. Error bars are obtained using an error matrix generated by 12 fit parameters.

η as small dips, and to the phase shift as small kinks. Indeed the CERN-Munich data [12] as shown in figure 2 may reveal some structure in the region $\sqrt{s} > 1.4\,\text{GeV}$, however the data are not accurate enough to be useful in distinguishing those higher mass poles. The data on η only exhibits $f_2(1270)$ clearly, so that fit to δ and η fails to reproduce other poles unambiguously. Nevertheless these higher mass poles do play a role in reducing the fit total width of f_2^{III}, meanwhile they play very little role in decreasing the total χ^2 and altering any other output. In the following we perform a 12 parameter fit to the CERN–Munich data on I=0 d wave $\pi\pi$ scattering phase shift up to 1.6 GeV. The 12 parameters consist of 6 relevant parameters and 6 'irrelevant' parameters. The former are: 1 for the virtual pole position, 2 for $f_2^{\text{II}}(1270)$ and 3 for $f_2^{\text{III}}(1270)$ (mass, total width and partial width into $\pi\pi$). The irrelevant parameters are the total widths and $BR(\pi\pi)$ of the 3 higher mass states. The extra resonances' masses are fixed at their values as provided by ref. [10]. The fit results are the following,

$$\begin{aligned}
\chi^2_{d.o.f.} &= 96.5/(80-12)\,; \\
M_v &= 1.73 \pm 0.04\,\text{MeV}\,; \\
M_{f_2}^{\text{II}} &= 1270.6 \pm 3.2\,\text{MeV}\,, \qquad \Gamma_{f_2}^{\text{II}} = 156.5 \pm 5.6\,\text{MeV}\,; \\
M_{f_2}^{\text{III}} &= 1278.6 \pm 22.2\,\text{MeV}\,, \qquad \Gamma_{f_2}^{\text{III}}(\text{tot}) = 225.0 \pm 76.2\,\text{MeV}\,; \\
BR(\pi\pi) &= 92.1 \pm 2.4\%\,; \\
a_2^0 &= (1.89 \pm 0.09) \times 10^{-3} m_\pi^{-4}\,, \qquad b_2^0 = (-2.00 \pm 0.02) \times 10^{-4} m_\pi^{-6}\,; \qquad (19)
\end{aligned}$$

where a_2^0 and b_2^0 are threshold parameters defined in eq. (9). We also provide in table 1 a few results of phase shift in order to compare with the results from ref. [13]. The results on additional resonances are not included in eq. (19), since they totally disagree with their PDG values.[2] Nevertheless it should be stressed that outputs as listed above are affected rather little with or without these additional resonances included. Especially a_2^0 is always in nice agreement with its χPT value, and b_2^0 is not. The reason why the b_2^0 parameter does not agree well with the χPT prediction may be that the b_2^0 parameter is related to higher order of momentum expansions and is therefore more sensitive to distant singularities. A numerical test is made and it is found that a distant second sheet pole with a very large

[2] The main reason for the failure is because a 3rd sheet narrow resonance's contribution to the phase shift and inelasticity as shown by figure 1 are rather tiny and concentrated in a small region of $\sqrt{s}$ which makes the fit heavily disturbed by the fluctuation of data. In order to get correct information about these 3rd sheet poles, it requires very dense data with very small error bars.

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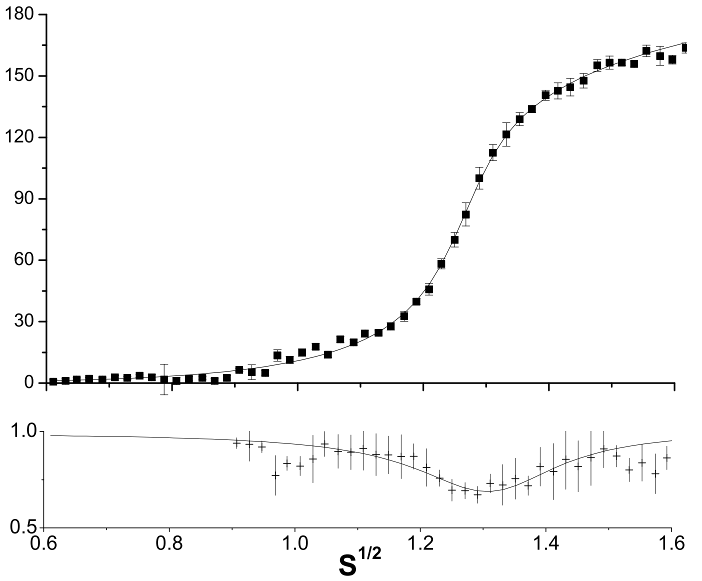

Figure 2: The fit on I=0 d-wave $\pi\pi$ scattering phase shift. Data are taken from ref. [12]. The higher resonances' effects are not included.

width can remedy the discrepancy on b_2^0 with very small influence to other outputs in eq. (19), but the situation does not correspond to a local χ^2 minimum and no meaningful conclusion can be drawn whether there is any physics behind this discrepancy.

The total χ^2 given by eq. (19) is not very good. From figure 2 one sees that the data fluctuation between 0.8 GeV and 1.1 GeV contributes a lot to the total χ^2. It is hard to solely ascribe the discrepancy to the data itself. Because in the energy region $K\bar{K}$ threshold opens and should contribute at least part of the discrepancy. The eq. (19) is only a narrow width approximation, and we know that one condition for the narrow width approximation to work is that the energy region is far away from any threshold. Therefore, the use of eq. (19) may also contribute to some of the discrepancy.

To conclude, within an appropriate parametrization form for the partial wave scattering S matrix incorporated with chiral symmetry. we have clearly demonstrated that the $f_2(1270)$ resonance exhibits a twin pole structure, i.e., it generates simultaneously both a 3rd sheet pole and a 2nd sheet pole. The two poles locate at similar position, though on different sheets. This is a picture exhibited by a typical couple-channel Breit-Wigner resonance: when the resonance's coupling to the elastic channel is dominant, two poles on different sheets coexist. In production experiments one sees the '3rd' sheet pole, whereas in phase shift analysis the nearly 180° jump is generated by the second sheet pole. The pole parameters of $f_2(1270)$ resonance are determined, the mass and width of the 3rd sheet pole, though contain rather large error bars, are found in agreement with the PDG value.

It is also worth noticing that our approach, for the first time, uncovers the existence of the virtual state from I=0 d-wave $\pi\pi$ scattering data. Actually the existence of such a virtual pole may be proved rigorously without rescuing to perturbation theory.[3] Our

[3] In fact there exists a virtual pole in every partial wave amplitude when $l \geq 2$. Such a phenomenon

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numerical analysis indicates that the location of the virtual pole is in good agreement with the χPT prediction. This is very remarkable, since the virtual state contributes tiny to the phase shift and it can hardly be seen directly from the data. The virtual pole definitely emerges, in the numerical analysis, as long as left hand cut integral provides a negative contribution when s is large, hence the conclusion is robust. Our work, to the best of our knowledge, is the first one in support of the theoretical prediction by analyzing experimental data.

Combining with our previous analysis on the virtual state in I=2 s-wave $\pi\pi$ scattering [2], we conclude that the perturbative chiral expansion in the small $|s|$ region is trustworthy, at least it provides a self-consistent and satisfactory description to the data, after embedding into a correctly unitarized parametrization form. This is not a trivial statement. For example the Padé approximation produces a singularity structure in total disagreement with χPT predictions, just not far below the two π threshold. [15] Therefore it is meaningful to reexamine more carefully the structure of the scattering amplitude in the small $|s|$ region.

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References

[1] V.E. Markushin, Z. Xiao and H.Q. Zheng, *Left hand singularities, hadron form factors and the properties of the sigma meson*, *Nucl. Phys.* **A 695** (2001) 273 [hep-ph/0011260].

[2] Z.Y. Zhou et al., *The pole structure of the unitary, crossing symmetric low energy $\pi\pi$ scattering amplitudes*, *JHEP* **02** (2005) 043 [hep-ph/0406271].

[3] G. Colangelo, J. Gasser and H. Leutwyler, *$\pi\pi$ scattering*, *Nucl. Phys.* **B 603** (2001) 125 [hep-ph/0103088].

[4] D. Aston et al., *A study of $K^-\pi^+$ scattering in the reaction $K^-p \to K^-\pi^+n$ at* 11 GeV/c, *Nucl. Phys.* **B 296** (1988) 493.

[5] H.Q. Zheng et al., *The kappa resonance in s wave πK scatterings*, *Nucl. Phys.* **A 733** (2004) 235 [hep-ph/0310293].

[6] Q. Ang et al., *A critical examination to the unitarized $\pi\pi$ scattering chiral amplitude*, *Commun. Theor. Phys.* **36** (2001) 563.

[7] H.q. Zheng, *How to parameterize a light and broad resonance (the sigma meson)*, hep-ph/0304173.

[8] Z.G. Xiao and H.Q. Zheng, *Analyticity and the N_c counting rule of s matrix poles*, hep-ph/0502199.

leads to the conclusion that there exists a weak essential singularity at $s = 0$ in full amplitude for $\pi^0\pi^0$ scattering, on the second sheet. [14]

[9] P.D.B. Collins, *An introduction to Regge theory and high energy physics* Cambridge University Press, 1977.

[10] Particle Data Group collaboration, S. Eidelman et al., *Review of particle physics*, *Phys. Lett.* **B 592** (2004) 1, (2004) and 2005 partial update for edtion 2006(URL: http://pdg.lbl.gov)

[11] J. Bijnens, et al., *Elastic $\pi\pi$ scattering to two loops*, *Phys. Lett.* **B 374** (1996) 210; J. Bijnens, G. Colangelo, G. Ecker, J. Gasser and M.E. Sainio, *Pion pion scattering at low energy*, *Nucl. Phys.* **B 508** (1997) 263 [hep-ph/9707291].

[12] B. Hyams et al., *$\pi\pi$ phase shift analysis from* 600 MeV *to* 1900 MeV, *Nucl. Phys.* **B 64** (1973) 134.

[13] J.L. Basdevant, C.D. Froggatt and J.L. Petersen, *Construction of phenomenological $\pi\pi$ amplitudes*, *Nucl. Phys.* **B 72** (1974) 413.

[14] A. Martin and F. Cheung, *Analyticity properties and bounds of the scattering amplitudes*, Gordon and Breach, New York 1970.

[15] G.Y. Qin, W.Z. Deng, Z.G. Xiao and H.Q. Zheng, *The [1,2] pade amplitudes for $\pi\pi$ scatterings in chiral perturbation theory*, *Phys. Lett.* **B 542** (2002) 89 [hep-ph/0205214].